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Generation of a four-qubit cluster state for atoms in a thermal cavity

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Abstract. We propose two schemes for generating a four-atom cluster state in a thermal cavity. With the assistant of a strong classical field the photon-number-dependent parts in the effective Hamiltonian are canceled. Thus the schemes are insensitive to the thermal field. The schemes can also be used to generate the cluster state for the trapped ions in thermal motion.

PACS. 03.67.Mn Entanglement production, characterization, and manipulation – 42.50.Dv Nonclassical states of the electromagnetic field, including entangled photon states; quantum state engineering and measurements – 42.50.Vk Mechanical effects of light on atoms, molecules, electrons, and ions

Quantum entanglement has attracted much attention. It not only gives the possibility for test of quantum mechanics against local hidden theory, but also is the basic physical resource for quantum computation and quantum communication. Experimentally the technology of preparing an entangled photon pair by spontaneous parametric down-conversion [1] is quite mature. four-photon entangled state [2] and five-photon entangled state [3] also have been obtained. On the other hand, recent advance in the cavity quantum electrodynamics system (cavity QED) and ion traps has opened the new prospects for quantum information processing. In cavity QED, a two-atom entangled state has been experimentally demonstrated by use of two Rydberg atoms crossing a nonresonant cavity [4]. The multi-atom Greenberger-Horne-Zeilinger (GHZ) states and W state also have been generated [5]. Some research groups also have realized a four-qubit GHZ state [6], a six-atom GHZ state [7] and eight-qubit W state [8] in ion traps. Recently Briegel et al. [9] introduced a class of entangled states, i.e., the cluster states. The cluster states have a high persistence of entanglement and can be regarded as a resource for GHZ states and are more immune to decoherence than GHZ states. On the other hand, cluster states have been shown to constitute a universal resource for quantum computation. The proof of Bell's theorem without the inequality was given for cluster states, and Bell inequality is maximally violated by four-qubit cluster state and is not violated by the four-qubit GHZ state. Zou et al. [10] have proposed a scheme for generation of polarization entangled cluster state. Zou et al. [11] and Dong et al. [12] have presented schemes for generating a four-atom cluster state via the resonant atom-cavity interaction, respectively. Cho et al. [13] also have shown a method for the generation of cluster states based on the cavity input-output process and the single-photon polarization measurement. Recently Zheng [14] has given two schemes for the generation of four-qubit cluster states in ion-trap systems. Here we propose two schemes to generate a cluster state of four atoms by the atom-cavity field interaction. The distinct advantage of the scheme is that during the passage of the atoms through the cavity field, a strong classical field is accompanied so that the photon-number-dependent parts are canceled. Thus the schemes are insensitive to both the cavity decay and the thermal field. More importantly, our second scheme can be carried out only by two steps and in this scheme the cluster state can be generated in much simpler manner than any of previous schemes, which might be highly important for the feasible experimental implementation.

We consider N identical two-level atoms simultaneously interacting with a single-mode cavity field, at the same time the atoms are driven by a classical field. In the rotating-wave approximation, the Hamiltonian for the system is [15–18]

$$H = \omega_0 S_z + \omega a^+ a + \sum_{j=1}^{N} \left[\frac{g}{2}(a^+ S_j^- + a S_j^+) + \frac{\Omega}{2}(S_j^+ e^{-i\omega_a t} + S_j^- e^{i\omega_a t}) \right] \quad (1)$$

where $S_z = (1/2) \sum_{j=1}^{N} (|e_j\rangle\langle e_j| - |g_j\rangle\langle g_j|)$, $s_j^+ = |e_j\rangle\langle g_j|$ and $s_j^- = |g_j\rangle\langle e_j|$, with $|e_j\rangle$ and $|g_j\rangle$ being the excited and ground states of the jth atom, a^+ and a are, respectively,

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the creation and annihilation operators for the cavity mode, g is the atom-cavity coupling strength, and δ is the detuning between the atomic transition frequency ω_0 and cavity frequency ω. ω_a is the frequency of the classical field. Assume $\omega_0 = \omega_a$, in the interaction picture, the interaction Hamiltonian is

$$H_I = \frac{\Omega}{2}\sum_{j-1}^{N}(S_j^+ + S_j^-) + \frac{g}{2}\sum_{j=1}^{N}(e^{-i\delta t}a^+S_j^- + e^{i\delta t}aS_j^+). \quad (2)$$

Define the new atomic basis [16,19]

$$|+_j\rangle \to \frac{1}{\sqrt{2}}(|g_j\rangle + |e_j\rangle), \quad (3)$$

$$|-_j\rangle \to \frac{1}{\sqrt{2}}(|g_j\rangle - |e_j\rangle). \quad (4)$$

The interaction Hamiltonian can be written as

$$H_I = \sum_{j-1}^{N}\left\{\Omega\sigma_{z,j} + \frac{g}{2}\left[e^{-i\delta t}a^+\left(\sigma_{z,j} + \frac{1}{2}\sigma_j^+ - \frac{1}{2}\sigma_j^-\right) + e^{-i\delta t}a\left(\sigma_{z,j} + \frac{1}{2}\sigma_j^- - \frac{1}{2}\sigma_j^+\right)\right]\right\}, \quad (5)$$

where $\sigma_{z,j} = (|+_j\rangle\langle +_j| - |-_j\rangle\langle -_j|)/2$, $\sigma_j^+ = |+_j\rangle\langle -_j|$ and $\sigma_j^- = |-_j\rangle\langle +_j|$.

According to Schrödinger's equation, the unitary transformation is

$$|\psi(t)\rangle = e^{-iH_0 t}\,|\psi'(t)\rangle\,, \quad (6)$$

where $H_0 = \sum_{j-1}^{N}\Omega\sigma_{z,j}$.

Then we have

$$i[d\,|\psi'(t)\rangle\,/dt] = H_I'\,|\psi'(t)\rangle\,, \quad (7)$$

where $H_I' = \sum_{j-1}^{N}\{(g/2)[e^{-i\delta t}a^+(\sigma_{z,j} + \sigma_j^+/2 - \sigma_j^-/2) + e^{-i\delta t}a(\sigma_{z,j} + \sigma_j^-/2 - \sigma_j^+/2)]\}$.

Assume that $\Omega \gg \delta$, g and can neglect the terms oscillating fast. Then the Hamiltonian can reduce to

$$H_I' = \frac{g}{2}\sum_{j=1}^{N}(e^{-i\delta t}a^+ + e^{i\delta t}a)\sigma_{z,j} = g(e^{-i\delta t}a^+ + e^{i\delta t}a)S_x, \quad (8)$$

where $S_x = (1/2)\sum_{j=1}^{N}(S_j^+ + S_j^-)$.

The evolution operator for Hamiltonian H_I' of equation (8) can be written in the form of [16,20]

$$U'(t) = e^{-iA(t)S_X^2 - iB(t)S_x a - iC(t)S_x a^+}.$$

By using the Schrödinger's equation

$$i\frac{dU'(t)}{dt} = H_i U'(t),$$

we have

$$B(t) = \int_0^t \frac{g}{2}e^{i\delta\tau}d\tau = \frac{g}{2i\delta}(e^{i\delta t} - 1),$$

$$C(t) = \int_0^t \frac{g}{2}e^{-i\delta\tau}d\tau = -\frac{g}{2i\delta}(e^{-i\delta t} - 1),$$

$$A(t) = i\int_0^t B(\tau)\frac{g}{2}e^{-i\delta\tau}d\tau = \frac{g^2}{4\delta}\left[t + \frac{1}{i\delta}(e^{-i\delta t} - 1)\right].$$

Assume $\delta t = 2\pi$, we obtain $B(t) = C(t) = 0$. Then we have

$$U'(t) = e^{-i\lambda t S_X^2}, \quad (9)$$

where $\lambda = g^2/4\delta$. The evolution operator of the system is given by

$$U(t) = e^{-iH_0 t}U'(t) = e^{-i\Omega t S_x - i\lambda t S_X^2}. \quad (10)$$

We note that the evolution operator of the Hamiltonian is independent of the cavity field state, allowing it to be in a thermal state.

Using the representation of the operator S_z, the atomic states $|g_1g_2...g_N\rangle$ and $|e_1e_2...e_N\rangle$ can be expressed as $|N/2, -N/2\rangle$ and $|N/2, N/2\rangle$. These state also be expanded in terms of the eigenstates of S_x [17]

$$|N/2, -N/2\rangle = \sum_{M=-N/2}^{N/2} C_M|N/2, M\rangle_x, \quad (11)$$

$$|N/2, N/2\rangle = \sum_{M=-N/2}^{N/2} C_M(-1)^{N/2-M}|N/2, M\rangle_x. \quad (12)$$

Assume that N atoms are initially in the state $|g_1g_2...g_N\rangle$, the evolution of the system is

$$|g_1g_2...g_N\rangle \xrightarrow{U(t)} \sum_{M=-N/2}^{N/2} C_M e^{-2i(\Omega M + \lambda M^2)t}\,|N/2, M\rangle_x\,. \quad (13)$$

Now we describe in detail how to generate a cluster state in cavity QED. Assume that three atoms are initially in the state $|g_1g_2g_3\rangle$. The evolution of the system is obtained by the above equation. Choose $\lambda t = \pi/4$ and $\Omega t = (2n + 3/4)\pi$, we have

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}e^{i(5/8)\pi}(|g_1g_2g_3\rangle + i\,|e_1e_2e_3\rangle). \quad (14)$$

Then we send atoms 1 and 2 across another single-mode cavity simultaneously. With the choice of $\lambda t = \pi/4$ and $\Omega t = 2n\pi$, we have the following transformation

$$|g_1g_2\rangle \to \frac{1}{\sqrt{2}}e^{-i\pi/4}(|g_1g_2\rangle - i\,|e_1e_2\rangle), \quad (15)$$

$$|e_1e_2\rangle \to \frac{1}{\sqrt{2}}e^{-i\pi/4}(|e_1e_2\rangle - i\,|g_1g_2\rangle). \quad (16)$$

Thus the state of the system will be transformed into

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}e^{i3\pi/8}(|g_1g_2g_3\rangle + i|e_1e_2e_3\rangle - i|e_1e_2g_3\rangle + |g_1g_2e_3\rangle). \quad (17)$$

Then let atom 3 cross a classical field tuned to the transition $|e\rangle \rightarrow |g\rangle$. Choose the amplitude and phase of the classical field appropriately so that this atom undergoes the following transition

$$|g_3\rangle \rightarrow \frac{1}{\sqrt{2}}(|g_3\rangle + |e_3\rangle), \tag{18}$$

$$|e_3\rangle \rightarrow \frac{1}{\sqrt{2}}(|g_3\rangle - |e_3\rangle). \tag{19}$$

Then we send atom 3 and atom 4, initially in the state $|g_4\rangle$, pass through a single-mode cavity simultaneously. After an interaction time $\lambda t = \pi/4$ and $\Omega t = 2(n+3/4)\pi$, and with a transformation $|g_3\rangle \rightarrow (1/\sqrt{2})(|g_3\rangle + i\,|e_3\rangle)$, $|e_3\rangle \rightarrow (1/\sqrt{2})(|g_3\rangle - i\,|e_3\rangle)$, we can obtain

$$|\psi_3\rangle = \frac{1}{2}e^{i5\pi/8}(|g_1g_2g_3g_4\rangle + |g_1g_2e_3e_4\rangle + |e_1e_2g_3g_4\rangle - |e_1e_2e_3e_4\rangle). \tag{20}$$

This state is just a four-atom cluster state.

Next we give another simpler scheme for generating a four-atom cluster state. Consider four atoms initially in the state $|g_1g_2g_3g_4\rangle$. We send the four atoms through a cavity simultaneously. The evolution of the system is obtained by equation (13). After an interaction time $\lambda t = \pi/4$ and $\Omega t = n\pi$, we obtain

$$|\psi\rangle_{1234} = \frac{1}{\sqrt{2}}e^{-i\pi/4}(|g_1g_2g_3g_4\rangle + i\,|e_1e_2e_3e_4\rangle). \tag{21}$$

Then we again send any two of four atoms, such as the atoms 1, 2 interact simultaneously with another cavity. Similarly after an interaction time $\lambda t = \pi/4$ and $\Omega t = 2n\pi$, the evolution of the state is given by equations (15) and (16), we have

$$|\psi\rangle_{1234} = \frac{1}{2}e^{-i\pi/2}(|g_1g_2g_3g_4\rangle + |g_1g_2e_3e_4\rangle - i\,|e_1e_2g_3g_4\rangle + i\,|e_1e_2e_3e_4\rangle). \tag{22}$$

Performing the transformation$|e_2\rangle \rightarrow i\,|e_2\rangle$, we can transform the state (20) into a four-atom cluster state

$$|\psi\rangle_{1234} = \frac{1}{2}e^{-i\pi/2}(|gggg\rangle + |ggee\rangle + |eegg\rangle - |eeee\rangle). \tag{23}$$

We note that the two schemes can also be applied to the ion trap system. We consider that N ions are confined in a linear trap. Then we simultaneously excite the ions with two lasers. If we tune the lasers sufficiently close to the sidebands, we can neglect all other vibrational modes, the Hamiltonian for the system is

$$H = \nu a^+ a + \omega_0 S_z + \{\Omega e^{-i\phi}\sum_{j=1}^{N} S_j^+ e^{i\eta(a^+ + a)} \times [e^{-i(\omega_0+\nu+\delta)t} + e^{-i(\omega_0-\nu-\delta)t]} + H.c.\},$$

where a^+ and a are the creation and annihilation operators for the collective vibrational mode, and $\eta = k/\sqrt{2\nu M}$ is the Lamb-Dicke parameter, with k being the wave vector along the trap axis and M is the mass of the ion collection. Here the lasers have the same Rabi frequency Ω, phase ϕ and wave vector k.

In the Lamb-Dicke regime, the interaction Hamiltonian in the interaction picture is [21]

$$H_i = i\eta\Omega e^{-i\phi}\sum_{j=1}^{2} S_j^+(a^+ e^{-i\delta t} + a e^{i\delta t}) + H.c. \tag{24}$$

As described by reference [17], the evolution operator of the effective Hamiltonian has same form as equation (9), with $\lambda = 2\eta^2\Omega^2/\delta$. Thus we can generate the entangled cluster states of four trapped ions using the procedure similar to that for cavity QED.

In conclusion, we have proposed two simple protocols to realize the generation of the four-atom cluster states in cavity QED. Comparing these two schemes, we find that the second one is always better than the first one, because the second one only includes two interactions of atoms with cavities and a single-qubit transformation to generate a four-atom cluster, but the first one must need more resource and more single-qubit rotation transformations to accomplish the same task. In cavity QED, the photon-number-dependent parts in the effective Hamiltonian are canceled with the assistance of a strong classical driving field. Thus our schemes are insensitive to both the cavity decay and thermal field. For the trapped ions, our schemes are insensitive to the thermal motion. They can be implemented by the present cavity QED and ion trap techniques.

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